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GENERAL SOLUTION OF THE EQUATIONS OF PARALLEL-FLOW MULTICHANNEL HEAT EXCHANGERS

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Abstract—The paper presents general solution of the systems of differential equations describing the distribution of temperature and temperature difference of fluids in exchanger channels. These equations have been derived in another paper [13]. It is found that for n channels the number of equations is at most n or $n - 1$. Solutions are given in a form enabling introduction of the boundary conditions. These solutions are obtained in the general case of channels forming a bundle and exchanging heat according to the principle of the maximum possible number of heat exchanges between channels and in most common practical cases in which the number of heat exchanges is reduced.

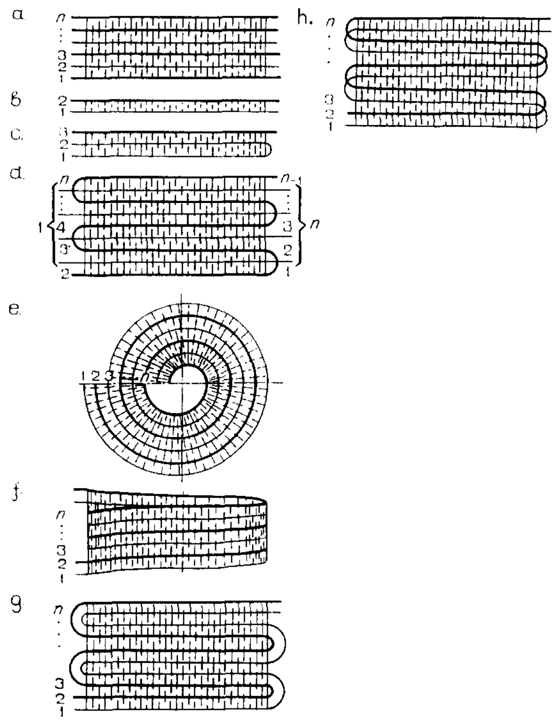
NOMENCLATURE

$a_1, a_2, \ldots a_n$,	integration constants;
a_{kl},	specific heat transference of the channel k for heat exchange between the channel k and l of the set $1, 2, \ldots n$;
$C_1, C_2, \ldots C_n$,	integration constants;
$D, D_1, D_2, \ldots D_n$,	determinants in Cramer's formulae;
$\mathscr{D}_1, \mathscr{D}_2, \ldots \mathscr{D}_n$,	determinants obtained from the determinants in Cramer's formulae;
i, j,	elements of the set $1, 2, \ldots n$ with the condition $i < j$ imposed on combination ij;
$i, (i)$,	number of roots of the characteristic equation;
k, l,	elements of the numerical set $1, 2, \ldots n$;
r_i $(= r_1, r_2 \ldots r_n)$,	integration constants constituting roots of the characteristic equation;
t_k $(= t_1, t_2, \ldots t_n)$,	local temperature of the fluid flowing in the channel k of the set $1, 2, \ldots n$;
x,	length, or annular co-ordinate of the channel;
$y_{ij} = t_i - t_j$,	temperature difference between fluids flowing in the channel i and j.

1. INTRODUCTION

PARALLEL-FLOW multichannel heat exchangers constitute a large group of heat exchangers. Figure 1 shows the most common practical types belonging to this group. Simple particular cases were considered by other authors [1–12]. A generalization of the methods for their solution to more complicated cases requires the solution of the general problem of heat exchange between n parallel channels forming a bundle as shown in Fig. 2. Such a general problem could be reduced to any particular case by introducing appropriate boundary conditions.

Longitudinal distribution of the temperature in the fluids $t_k(x)$ or the distribution of the temperature difference $y_{kl}(x) = t_k(x) - t_l(x)$ between two channels k and l in a bundle of n heat exchanging channels are described by systems of homogeneous linear differential equations



Explanations:

1. Continuous lines denote the heat exchanging flows with any flow directions.
2. The areas shaded with dashed lines denote the considered heat-transfer regions. The direction of the dashed lines indicates the direction of heat flow across the channel walls considered.
3. The brackets express the fact that the flows are mixed so that temperature is uniform in any cross section of the flows taken in brackets.

(a) Multifluid heat exchanger.

(b) Ordinary parallel flow.

(c) Field tube.

(d) Multiloop, or in other words multipass shell heat exchanger.

(e) Spiral heat exchanger (according to the Swedish Rosenblad patents, for instance).

(f) Screw heat exchanger (a new concept of a high efficiency heat exchanger).

(g) Wave-formed noncross heat exchanger (in line or concentric).

(h) Wave-formed cross heat exchanger (in line or concentric) giving a higher efficiency in comparison with last heat exchanger.

FIG. 1. Schematic drawing of more important types of parallel-flow multichannel heat exchangers.

of the first order [13]. These equations have been derived under the assumption of maximum possible number of heat exchanges between channels and the conventional assumption of the theory of heat exchangers. There are:

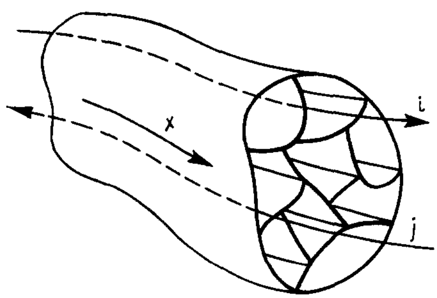


FIG. 2. General case of parallel-flow multichannel heat exchanger in the form of a bundle of channels.

1. The heat-transfer process is stationary.
2. The walls of the channels do not conduct heat in the direction of the axis of the channels.
3. The heat-transfer surfaces separating fluids flowing in channels k and l are of equal perimeters.

The equations just mentioned are for t_k

$$\left.\begin{aligned}
&\frac{dt_1}{dx} + t_1\sum_{k=1}^{n} a_{1k} - \sum_{k=1}^{n} a_{1k}t_k = 0,\\
&\frac{dt_2}{dx} + t_2\sum_{k=1}^{n} a_{2k} - \sum_{k=1}^{n} a_{2k}t_k = 0,\\
&\dots\dots\dots\dots\dots\dots\\
&\frac{dt_n}{dx} + t_n\sum_{k=1}^{n} a_{nk} - \sum_{k=1}^{n} a_{nk}t_k = 0.
\end{aligned}\right\} \tag{1}$$

This is a system of n equations with n unknown functions $t_1, t_2, \dots t_n$. In general, we can write

$$\frac{dt_k}{dx} + t_k\sum_{k=1}^{n} a_{kl} - \sum_{k=1}^{n} a_{kl}t_l = 0. \tag{1a}$$

Next for y_{kl} we have

$$\left.\begin{aligned}
&\frac{dy_{11}}{dx} + \sum_{k=1}^{n} a_{1k}\,y_{1k} - \sum_{k=1}^{n} a_{1k}\,y_{1k} = 0, \dots \frac{dy_{1n}}{dx} + \sum_{k=1}^{n} a_{1k}\,y_{1k} - \sum_{k=1}^{n} a_{nk}\,y_{nk} = 0,\\
&\frac{dy_{21}}{dx} + \sum_{k=1}^{n} a_{2k}\,y_{2k} - \sum_{k=1}^{n} a_{1k}\,y_{1k} = 0, \dots \frac{dy_{2n}}{dx} + \sum_{k=1}^{n} a_{2k}\,y_{2k} - \sum_{k=1}^{n} a_{nk}\,y_{nk} = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{dy_{n1}}{dx} + \sum_{k=1}^{n} a_{nk}\,y_{nk} - \sum_{k=1}^{n} a_{1k}\,y_{1k} = 0, \dots \frac{dy_{nn}}{dx} + \sum_{k=1}^{n} a_{nk}\,y_{nk} - \sum_{k=1}^{n} a_{nk}\,y_{nk} = 0.
\end{aligned}\right\} \tag{2}$$

In these equations k, l are any two elements of the set 1, 2, . . . n and a_{kl} is a specific heat transference* of the channel k for heat exchange with the channel l. The set of a_{kl} constitutes a square matrix $[a_{kl}]$. The specific heat transference is defined as $a_{kl} = k_{kl}h_{kl}/W_k$, where $k_{kl} = k_{lk}$ is heat-transfer coefficient between the channel k and l, $h_{kl} = h_{lk}$ is the common perimeter of channels and W_k is a water equivalent of the fluid flowing in the channel k.

The heat transference represents the ratio of the heat that transfers across the wall of a channel to the heat flowing along the wall. Therefore it characterizes the capability of the channel to exchange heat with another channel or the ambient medium. Its value is 0 for adiabatic flow and ∞ for perfect non-adiabatic flow.

The sign of W_k and hence of a_{kl} depends on the flow direction. The equations have been written for the same flow directions according to the direction of the co-ordinate x. Therefore for channels with opposite flow directions the sign of a_{kl} must be changed. Of course, $a_{kk} = 0$. We assume that the remaining $a_{kl} \neq a_{lk}$ are not functions of x or t_k and that they are constant. Hence equations (1) and (2) have in our considerations constant coefficients.

Instead of k, l we can use the symbols i, j with the condition $i < j$ for elements of the set 1, 2, . . . n. Then equations (2) which are elements of the square matrix, can be divided into three systems. In general we can write

$$\frac{dy_{ij}}{dx} + \sum_{k=1}^{n} a_{ik}\,y_{ik} - \sum_{k=1}^{n} a_{jk}\,y_{jk} = 0 \tag{2a}$$

$$\frac{dy_{ii}}{dx} + \sum_{k=1}^{n} a_{ik}\,y_{ik} - \sum_{k=1}^{n} a_{ik}\,y_{ik} = 0 \tag{2b}$$

* In other words this is a number of heat-transfer units per unit linear or angular length of a channel k. The number of heat-transfer units of the channel is denoted often by NTU [W. M. KAYS and A. L. LONDON, *Compact Heat Exchangers*. McGraw-Hill, New York (1958)].

$$\frac{dy_{ji}}{dx}+\sum_{k=1}^{n}a_{jk}\,y_{jk}-\sum_{k=1}^{n}a_{ik}\,y_{ik}=0. \tag{2c}$$

System (2a) contains the equations placed above the main diagonal of equation matrix (2). The system (2b) contains the equations on the main diagonal and has trivial zero solutions. The system (2c) contains the equations placed below the main diagonal. Since $y_{ij}=-y_{ji}$ equations (2c) are identical with equation (2a) and we can consider only $(n^2-n)/2$ equations (2a) or (2c) for dy_{ij}/dx or dy_{ji}/dx.

2. GENERAL SOLUTIONS OF THE SYSTEM OF EQUATIONS FOR t_k

System (1) can be reduced to the Cauchy normal form by omitting the terms containing the factors $a_{ii}=0$ in the second sums and by ordering the equations according to the subscripts $1, 2, \ldots n$ of t.

$$\left.\begin{aligned}
\frac{dt_1}{dx}&=-\sum_{k=1}^{n}a_{1k}\,t_1+a_{12}\,t_2+\ldots\quad a_{1n}\,t_n,\\
\frac{dt_2}{dx}&=a_{21}\,t_1-\sum_{k=1}^{n}a_{2k}\,t_2+\ldots\quad a_{2n}\,t_n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\frac{dt_n}{dx}&=a_{n1}\,t_1+a_{n2}\,t_2+\ldots-\sum_{k=1}^{n}a_{nk}\,t_n.
\end{aligned}\right\} \tag{3}$$

The functions depending linearly on their derivatives can be only exponential ones. Therefore a particular solution (i) of system (1) can be written in the form

$$t_1^{(i)}=\alpha_1^{(i)}\,e^{r_i x},\qquad t_2=\alpha_2^{(i)}\,e^{r_i x},\ldots t_n=\alpha_n^{(i)}\,e^{r_i x}. \tag{4}$$

Functions (4) substituted in equations (3) give a set of homogeneous linear algebraic equations of the first power.

$$\left.\begin{aligned}
-(r_i+\sum_{k=1}^{n}a_{1k})\,\alpha_1^{(i)}+a_{12}\,\alpha_2^{(i)}+\ldots\quad a_{1n}\,\alpha_n^{(i)}&=0,\\
a_{21}\,\alpha_1^{(i)}-(r_i+\sum_{k=1}^{n}a_{2k})\,\alpha_2^{(i)}+\ldots\quad a_{2n}\,\alpha_n^{(i)}&=0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\
a_{n1}\,\alpha_1^{(i)}+a_{n2}\,\alpha_2^{(i)}+\ldots-(r_i+\sum_{k=1}^{n}a_{nk})\,\alpha_n^{(i)}&=0.
\end{aligned}\right\} \tag{5}$$

This set determines the constants $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots \alpha_n^{(i)}$ in terms of one of them at least $\alpha_k^{(i)}$ which should be chosen as an arbitrary constant, in function of the constants r_i and prescribed coefficients a_{kl}.

It is known from the theory of algebraic equations that set (5) has at least one non-zero solution for $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots \alpha_n^{(i)}$ if its characteristic determinant is zero.

$$\begin{vmatrix}
-(r_i+\sum_{k=1}^{n}a_{1k}) & a_{12}\ldots & a_{1n}\\
a_{21} & -(r_i+\sum_{k=1}^{n}a_{2k})\ldots & a_{2n}\\
\ldots & \ldots & \ldots\\
a_{n1} & a_{n2}\ldots & -(r_i+\sum_{k=1}^{n}a_{nk})
\end{vmatrix}=0. \tag{6}$$

This condition constitutes the characteristic equation of system (3) and determines the constants r_i which are the roots of this equation. The number of these roots is n because equation (6) is of degree n in r_i. Thus functions (4) are particular solutions of equation (3) for $i = 1, 2, \dots n$.

To obtain the general solution of the system (3) it is necessary to verify first, whether the roots are simple or multiple ones. We can exclude at once as non-interesting from the physical point of view the case of multiple zero-roots giving particular solutions (4) in the form of constants $\alpha_1^{(i)}$, $\alpha_2^{(i)}, \dots \alpha_n^{(i)}$.

The question of these roots is answered in a general manner by the following theorem.

Theorem 1.

If $a_{ii} = 0$, *then equation* (6) *has simple roots only.*

Proof

Let us transform determinant (6) in the following manner, for instance:

We add to column 1 all the other columns.

We cancel the coefficients $a_{ii} = 0$ occurring in the first column.

We subtract row 1 from all the other rows.

We expand the determinant thus obtained with respect to the first column.

Since $(-1)^3 \neq 0$, we obtain

$$r_i \begin{vmatrix} -(r_i + \sum_{k=1}^{n} a_{2k} + a_{12}) & a_{23} - a_{13} \dots & a_{2n} - a_{1n} \\ a_{32} - a_{12} & -(r_i + \sum_{k=1}^{n} a_{3k} + a_{13}) \dots & a_{3n} - a_{1n} \\ \dots & \dots & \dots \\ a_{n2} - a_{12} & a_{n3} - a_{13} \dots & -(r_i + \sum_{k=1}^{n} a_{nk} + a_{1n}) \end{vmatrix} = 0. \quad (6a)$$

It is seen that r_i is a factor of determinant (6a). This means that for $a_{ii} = 0$ the constant term of equation (6) is zero.* This equation has therefore zero-root $r_n = 0$.

Other roots are identical with the roots of non-homogeneous equation (6a) of the power $n - 1$ obtained from homogeneous equation (6) of the degree n.

We observe that determinant (6a) of the order $n - 1$ is different from every minor of the order $n - 1$ of determinant (6). Therefore the remaining roots cannot satisfy simultaneously different algebraic equations with different coefficients and all these minors must be different from zero and linearly independent.

Now we can consider the values of the first derivative of equation (6) for $r_i = r_1, r_2, \dots r_n$. Differentiating determinant (6) we have

* This result can be obtained in a different manner on the basis of the properties of the characteristic polynomials of a square matrix.

$$
\begin{vmatrix}
-(r_i + \sum_{k=1}^{n} a_{2k}) & a_{23} & \dots & a_{2n} \\
a_{32} & -(r_i + \sum_{k=1}^{n} a_{3k}) & \dots & a_{3n} \\
\dots & \dots & \dots & \dots \\
a_{n2} & a_{n3} & \dots & -(r_i + \sum_{k=1}^{n} a_{nk})
\end{vmatrix}
$$

$$
+ \begin{vmatrix}
-(r_i + \sum_{k=1}^{n} a_{1k}) & a_{13} & \dots & a_{1n} \\
a_{31} & -(r_i + \sum_{k=1}^{n} a_{3k}) & \dots & a_{3n} \\
\dots & \dots & \dots & \dots \\
a_{n1} & a_{n3} & \dots & -(r_i + \sum_{k=1}^{n} a_{nk})
\end{vmatrix}
$$

$$
\dots + \begin{vmatrix}
-(r_i + \sum_{k=1}^{n} a_{1k}) & a_{12} & \dots & a_{1,n-1} \\
a_{21} & -(r_i + \sum_{k=1}^{n} a_{2k}) & \dots & a_{2,n-1} \\
\dots & \dots & \dots & \dots \\
a_{n-1,1} & a_{n-1,2} & \dots & -(r_i + \sum_{k=1}^{n} a_{n-1,k})
\end{vmatrix}.
$$

This derivative is the sum of the n principal minors, of the order $n - 1$ of determinant (6) and is a polynomial of the degree $n - 1$ in r_i.

We must consider the possibility of three cases:

1. The derivative is zero and all the summands of the sum are zero.
2. The derivative is zero and not all the summands are zero.
3. The derivative is different from zero and not all the summands are zero.

Case 1 is impossible because the derivative is a linear combination of minors. Each of them have been found to be different from zero for all r_i.

Case 2 constitutes a singularity of case 3 and cannot influence the character of r_i because the derivative is the linear combination of equations which are not satisfied for all r_i.

Case 3 is thus a general one. Hence it follows that the roots of equation (6) are simple.*

On the basis of Theorem 1 we find that $n - 1$ equations of set (5) are linearly independent and only one constant $\alpha_k^{(i)}$ can be chosen arbitrary. The remaining constants $\alpha_1^{(i)}, \alpha_2^{(i)}, \dots \alpha_{k-1}^{(i)}, \alpha_{k+1}^{(i)} \dots \alpha_n^{(i)}$ can be expressed in terms of it.

Denoting by $D^{(i)}, D_1^{(i)}, D_2^{(i)}, \dots D_{k-1}^{(i)}, D_{k+1}^{(i)}, \dots D_n^{(i)}$ the determinants in Cramer's formulae obtained from the minors of the order $n - 1$ of determinant (6) we can express the constants $\alpha_1^{(i)}$, $\alpha_2^{(i)}, \dots \alpha_n^{(i)}$ by

* Paper [14] demonstrates the validity of Theorem 1 although for solving simple problems it is not necessary to use it.

$$\left.\begin{aligned}
\alpha_1^{(i)} &= \frac{D_1^{(i)}}{D^{(i)}} = \alpha_k^{(i)} \frac{\mathscr{D}_1^{(i)}}{D^{(i)}},\\
\alpha_2^{(i)} &= \frac{D_2^{(i)}}{D^{(i)}} = \alpha_k^{(i)} \frac{\mathscr{D}_2^{(i)}}{D^{(i)}},\\
&\dots\dots\dots\dots\\
\alpha_{k-1}^{(i)} &= \frac{D_{k-1}^{(i)}}{D^{(i)}} = \alpha_k^{(i)} \frac{\mathscr{D}_{k-1}^{(i)}}{D^{(i)}},\\
\alpha_k^{(i)} &= \text{arbitrary constant},\\
\alpha_{k+1}^{(i)} &= \frac{D_{k+1}^{(i)}}{D^{(i)}} = \alpha_k^{(i)} \frac{\mathscr{D}_{k+1}^{(i)}}{D^{(i)}}\\
&\dots\dots\dots\dots\\
\alpha_n^{(i)} &= \frac{D_n^{(i)}}{D^{(i)}} = \alpha_k^{(i)} \frac{\mathscr{D}_n^{(i)}}{D^{(i)}}
\end{aligned}\right\} \tag{7}$$

where, as is known, $(i) = 1, 2, \dots n$ are superscripts denoting the roots of equation (6). The determinants $\mathscr{D}_1^{(i)}, \mathscr{D}_2^{(i)}, \dots \mathscr{D}_{k-1}^{(i)}, \mathscr{D}_{k+1}^{(i)}, \dots \mathscr{D}_n^{(i)}$ are obtained from the determinants $D_1^{(i)}$, $D_2^{(i)}, \dots D_{k-1}^{(i)}, D_{k+1}^{(i)}, \dots D_n^{(i)}$ by rejecting the constant factors $\alpha_k^{(i)}$ $(= \alpha_k^{(1)}, \alpha_k^{(2)}, \dots \alpha_k^{(n)})$ occurring in successive columns $1, 2, \dots n$.

The determinant $D^{(i)}$ is obtained by cancelling one column k and one row of determinant (6). This column and row can be optional, because all the minors of determinant (6) are non-zero as follows from Theorem 1. The determinants $\mathscr{D}_1^{(i)}, \mathscr{D}_2^{(i)}, \dots \mathscr{D}_{k-1}^{(i)}, \mathscr{D}_{k+1}^{(i)}, \dots \mathscr{D}_n^{(i)}$ are obtained from the minor $D^{(i)}$ by replacing its successive columns with the cancelled column k with opposite sign.

It is known that the set of all $n = i_{\max}$ linearly independent particular solutions in the form

$$\begin{aligned}
&t_1^{(1)},\ t_2^{(1)}, \dots t_n^{(1)}, \text{ (the first solution)}\\
&t_1^{(2)},\ t_2^{(2)}, \dots t_n^{(2)}, \text{ (the second solution)}\\
&\dots\dots\dots\\
&t_1^{(n)},\ t_2^{(n)}, \dots t_n^{(n)}, \text{ (the } n\text{th solution)}
\end{aligned}$$

where

$$\begin{aligned}
t_1^{(1)} &= \alpha_1^{(1)} e^{r_1 x}, & t_2^{(1)} &= \alpha_2^{(1)} e^{r_1 x}, \dots & t_n^{(1)} &= \alpha_n^{(1)} e^{r_1 x},\\
t_1^{(2)} &= \alpha_1^{(2)} e^{r_2 x}, & t_2^{(2)} &= \alpha_2^{(2)} e^{r_2 x}, \dots & t_n^{(2)} &= \alpha_n^{(2)} e^{r_2 x},\\
&\dots\dots\dots\dots\dots\dots\\
t_1^{(n)} &= \alpha_1^{(n)} e^{r_n x}, & t_2^{(n)} &= \alpha_2^{(n)} e^{r_n x}, \dots & t_n^{(n)} &= \alpha_n^{(n)} e^{r_n x},
\end{aligned}$$

determines the general solution in the form of the following set of linear combinations of particular solutions

$$\left.\begin{aligned}
t_1 &= C_1 t_1^{(1)} + C_2 t_1^{(2)} + \dots C_n t_1^{(n)},\\
t_2 &= C_1 t_2^{(1)} + C_2 t_2^{(2)} + \dots C_n t_2^{(n)},\\
&\dots\dots\dots\dots\\
t_n &= C_1 t_n^{(1)} + C_2 t_n^{(2)} + \dots C_n t_n^{(n)}.
\end{aligned}\right\} \tag{8}$$

This solution can be written in a form more precise and more convenient for practical applications. Using the formulae (7) determining the constants $a_1^{(i)}, a_2^{(i)}, \dots a_n^{(i)}$ we obtain

$$\left.\begin{aligned}
t_1 &= C_1\, a_k^{(1)} \frac{\mathscr{D}_1^{(1)}}{D^{(1)}} e^{r_1 x} + C_2\, a_k^{(2)} \frac{\mathscr{D}_1^{(2)}}{D^{(2)}} e^{r_2 x} + \dots C_n\, a_k^{(n)} \frac{\mathscr{D}_1^{(n)}}{D^{(n)}} e^{r_n x},\\
t_2 &= C_1\, a_k^{(1)} \frac{\mathscr{D}_2^{(1)}}{D^{(1)}} e^{r_1 x} + C_2\, a_k^{(2)} \frac{\mathscr{D}_2^{(2)}}{D^{(2)}} e^{r_2 x} + \dots C_n\, a_k^{(n)} \frac{\mathscr{D}_2^{(n)}}{D^{(n)}} e^{r_n x},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
t_{k-1} &= C_1\, a_k^{(1)} \frac{\mathscr{D}_{k-1}^{(1)}}{D^{(1)}} e^{r_1 x} + C_2\, a_k^{(2)} \frac{\mathscr{D}_{k-1}^{(2)}}{D^{(2)}} e^{r_2 x} + \dots C_n\, a_k^{(n)} \frac{\mathscr{D}_{k-1}^{(n)}}{D^{(n)}} e^{r_n x},\\
t_k &= C_1\, a_k^{(1)} e^{r_1 x} + C_2\, a_k^{(2)} e^{r_2 x} + \dots C_n\, a_k^{(n)} e^{r_n x},\\
t_{k+1} &= C_1\, a_k^{(1)} \frac{\mathscr{D}_{k+1}^{(1)}}{D^{(1)}} e^{r_1 x} + C_2\, a_k^{(2)} \frac{\mathscr{D}_{k+1}^{(2)}}{D^{(2)}} e^{r_2 x} + \dots C_n\, a_k^{(n)} \frac{\mathscr{D}_{k+1}^{(n)}}{D^{(n)}} e^{r_n x},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
t_n &= C_1\, a_k^{(1)} \frac{\mathscr{D}_n^{(1)}}{D^{(1)}} e^{r_1 x} + C_2\, a_k^{(2)} \frac{\mathscr{D}_n^{(2)}}{D^{(2)}} e^{r_2 x} + \dots C_n\, a_k^{(n)} \frac{\mathscr{D}_n^{(n)}}{D^{(n)}} e^{r_n x}.
\end{aligned}\right\} \quad (9)$$

The arbitrary constants $a_k^{(i)}$ $(= a_k^{(1)}, a_k^{(2)}, \dots a_k^{(n)})$ which are different for different (i), can be included in the constants $C_1, C_2, \dots C_n$, viz. $a_k^{(1)}$ in C_1, $a_k^{(2)}$ in $C_2, \dots a_k^{(n)}$ in C_n.

Moreover, from the form of determinant (6) it is evident, that the minors $D^{(1)}, D^{(2)}, \dots D^{(n)}$ are exclusively functions of the constants r_i and of the coefficients a_{kl} which are prescribed. Hence they can be regarded as constant values independent of x or t_k. Therefore it is possible for the denominators $D^{(1)}, D^{(2)}, \dots D^{(n)}$ to be included in the same constants $C_1, C_2, \dots C_n$, viz. $D^{(1)}$ in C_1, $D^{(2)}$ in $C_2, \dots D^{(n)}$ in C_n.

Thus general solution (9) can be written in a simplified form containing the constants C_1, $C_2, \dots C_n$ only

$$\left.\begin{aligned}
t_1 &= C_1\, \mathscr{D}_1^{(1)} e^{r_1 x} + C_2\, \mathscr{D}_1^{(2)} e^{r_1 x} + \dots C_n \mathscr{D}_1^{(n)} e^{r_n x},\\
t_2 &= C_1\, \mathscr{D}_2^{(1)} e^{r_1 x} + C_1\, \mathscr{D}_2^{(2)} e^{r_2 x} + \dots C_n\, \mathscr{D}_2^{(n)} e^{r_n x},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
t_{k-1} &= C_1\, \mathscr{D}_{k-1}^{(1)} e^{r_1 x} + C_2\, \mathscr{D}_{k-1}^{(2)} e^{r_2 x} + \dots C_n\, \mathscr{D}_{k-1}^{(n)} e^{r_n x},\\
t_k &= C_1 D^{(1)} e^{r_1 x} + C_2\, D^{(2)} e^{r_2 x} + \dots C_n\, D^{(n)}\, e^{r_n x},\\
t_{k+1} &= C_1\, \mathscr{D}_{k+1}^{(1)} e^{r_1 x} + C_2\, \mathscr{D}_{k+1}^{(2)} e^{r_2 x} + \dots C_n\, \mathscr{D}_{k+1}^{(n)} e^{r_n x},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
t_n &= C_1\, \mathscr{D}_n^{(1)} e^{r_1 x} + C_2\, \mathscr{D}_n^{(2)} e^{r_2 x} + \dots C_n\, \mathscr{D}_n^{(n)} e^{r_n x}.
\end{aligned}\right\} \quad (10)$$

In solution (10) the constants $C_1, C_2, \dots C_n$ are unknown. It is possible to determine them by writing general solution (10) for the given boundary conditions, reordering the equations according to $C_1, C_2, \dots C_n$ and solving the set of non-homogeneous linear algebraic equations of the first power.

Introducing the values of $C_1, C_2, \dots C_n$, thus obtained, into general solution (10), we otbain the solution which corresponds exactly to the conditions and the character of the problem.

The solution will be the set of algebraic formulae describing the temperature distributions $t_1(x)$, $t(x)$, . . . $t_n(x)$ required.

Sometimes, for example in the case of a multifluid heat exchanger, which is shown in Fig. 1a and in the general case of a channel-bundle shown in Fig. 2, where the boundary temperatures $t_1, t_2, \ldots t_n$ for $x = 0$ or $x = s$ are given, the set of equations (10) need no such rearrangement.

Form (10) of the general solution is more convenient than (9) because we avoid the n arbitrary constants $\alpha_k^{(1)}, \alpha_k^{(2)}, \ldots \alpha_k^{(n)}$ which vanish. In addition we avoid n^2 operations of division of two determinants.

3. GENERAL SOLUTION OF SYSTEMS OF EQUATIONS FOR y_{ij}

In Section 1 it has been mentioned that from matrix (2) of n^2 equations only $(n^2 - n)/2$ equations can be used. These are either those on the upper or the lower side of the main diagonal. Thus, in the general case of a multichannel heat exchanger with maximum possible heat exchanges we must use the complete number of the $(n^2 - n)/2$ equations determining all the y_{ij}.

However, according to the definition $y_{ij} = t_i - t_j$ we have

$$y_{ij} = y_{i,i+1} + y_{i+1,i+2} + \ldots y_{j-1,j}. \tag{11}$$

Hence it is easy to observe in equation (2) and especially in equation (2a) that all the equations for $\mathrm{d}y_{ij}/\mathrm{d}x$ with $j > i + 1$ are sums of the $j - i$ corresponding equations for $\mathrm{d}y_{k,\ k+1}/\mathrm{d}x$ with k in the interval $i \leqslant k \leqslant j - 1$. Therefore the equations with $j > i + 1$ do not express any other relations in the system of equations (2a) than those with $j = i + 1$ and are not needed for the solution. Thus the distribution of the temperature difference between any two channels of a bundle containing n channels can be determined by solving the following system of $n - 1$ differential equations for $\mathrm{d}y_{k,k+1}/\mathrm{d}x$

$$\left.\begin{array}{l}
\dfrac{\mathrm{d}y_{12}}{\mathrm{d}x} + \sum\limits_{k=1}^{n} a_{1k}\, y_{1k} - \sum\limits_{k=1}^{n} a_{2k}\, y_{2k} = 0, \\
\dfrac{\mathrm{d}y_{23}}{\mathrm{d}x} + \sum\limits_{k=1}^{n} a_{2k}\, y_{2k} - \sum\limits_{k=1}^{n} a_{3k}\, y_{3k} = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\dfrac{\mathrm{d}y_{n-1,n}}{\mathrm{d}x} + \sum\limits_{k=1}^{n} a_{n-1,k}\, y_{n-1,k} - \sum\limits_{k=1}^{n} a_{nk}\, y_{nk} = 0,
\end{array}\right\} \tag{12}$$

and evaluating the remaining functions y_{ij} with $j > i + 1$ according to formula (11).

System (12) consists of the equations located directly above the main diagonal of matrix (2). This system is indefinite and can be transformed into a definite one, for the functions $y_{ij} = y_{12}$, $y_{23}, \ldots y_{n-1,n}$ in which the indices j, i denote pairs of neighbouring elements of the set 1, 2,. . . n.

To perform this transformation we must express all the equations of system (12) using the following substitution

$$\frac{\mathrm{d}y_{i,i+1}}{\mathrm{d}x} + \sum_{k=1}^{n} a_{ik}\, y_{ik} - \sum_{k=1}^{n} a_{\,i+1,k}\, y_{i+1,k}$$

$$= \frac{\mathrm{d}y_{i,i+1}}{\mathrm{d}x} + (a_{i1} y_{i1} + a_{i2} y_{i2} + \ldots a_{ii} y_{ii} + a_{i,i+1} y_{i,i+1} + a_{i,i+2} y_{i,i+2} + \ldots a_{i,n-1} y_{i,n-1} + a_{in} y_{in})$$
$$+ (a_{i+1,1} y_{i+1,1} + a_{i+1,2} y_{i+1,2} + \ldots a_{i+1,i-1} y_{i+1,i-1} + a_{i+1,i} y_{i+1,i} + a_{i+1,i+1} y_{i+1,i+1}$$
$$+ \ldots a_{i+1,n-1} y_{i+1,n-1} + a_{i+1,n} y_{i+1,n})$$

and on introducing equation (11), we obtain

$$
= \frac{dy_{i,i+1}}{dx} + \left\{
\begin{array}{lll}
a_{i1} & (y_{i,i-1} & + y_{i-1,i-2} + \dots y_{32} + y_{21}) \\
+ a_{i2} & (y_{i,i-1} & + y_{i-1,i-2} + \dots y_{32}) \\
\dots\dots & & \\
+ a_{ii} & (y_{ii}) & \\
+ a_{i,i+1} & (y_{i,i+1}) & \\
+ a_{i,i+2} & (y_{i,i+1} & + y_{i+1,i+2}) \\
\dots\dots & & \\
+ a_{i,n-1} & (y_{i,i+1} & + y_{i+1,i+2} + \dots y_{n-2,n-1}) \\
+ a_{i,n} & (y_{i,i+1} & + y_{i+1,i+2} + \dots y_{n-2,n-1} + y_{n-1,n})
\end{array}
\right\}
$$

$$
- \left\{
\begin{array}{llll}
a_{i+1,1} & (y_{i+1,i} & + y_{i,i-1} & + \dots y_{32} + y_{21}) \\
+ a_{i+1,2} & (y_{i+1,i} & + y_{i,i-1} & + \dots y_{32}) \\
\dots\dots & & & \\
+ a_{i+1,i-1} & (y_{i+1,i} & + y_{i,i-1}) & + \\
+ a_{i+1,i} & (y_{i+1,i}) & & \\
+ a_{i+1,i+1} & (y_{i+1,i+1}) & & \\
\dots\dots & & & \\
+ a_{i+1,n-1} & (y_{i+1,i+2} & + y_{i+2,i+3} & + \dots y_{n-2,n-1}) \\
+ a_{i+1,n} & (y_{i+1,i+2} & + y_{i+2,i+3} & + \dots y_{n-2,n-1} + y_{n-1,n})
\end{array}
\right\}
$$

Next, ordering according to y_{12}, $y_{23} \dots y_{n-1,n}$ and taking into account that $y_{ij} = -y_{ji}$ and $a_{ii} = 0$, $a_{i+1,i+1} = 0$ we have

$$
\left.
\begin{aligned}
&\frac{dy_{i,i+1}}{dx} + \sum_{k=1}^{n} a_{ik}\, y_{ik} - \sum_{k=1}^{n} a_{i+1,k}\, y_{i+1,k} \\
&= \frac{dy_{i,i+1}}{dx} + \sum_{k=1}^{1} (a_{i+1,k} - a_{ik})\, y_{12} + \sum_{k=1}^{2} (a_{i+1,k} - a_{ik})\, y_{23} \dots \\
&+ \sum_{k=1}^{i-1} (a_{i+1,k} - a_{ik})\, y_{i-1,i} + \Big(\sum_{k=1}^{i} a_{i+1,k} + \sum_{k=i+1}^{n} a_{ik}\Big)\, y_{i,i+1} + \sum_{k=i+2}^{n} (a_{ik} - a_{i+1,k})\, y_{i+1,i+2} \dots \\
&+ \sum_{k=n-1}^{n} (a_{ik} - a_{i+1,k})\, y_{n-2,n-1} + \sum_{k=n}^{n} (a_{ik} - a_{i+1,k})\, y_{n-1,n}.
\end{aligned}
\right\} \quad (13)
$$

Now we express all the equations of system (12) by expansion (13) in the functions y_{12}, $y_{23}, \dots y_{n-1,n}$. After ordering the coefficients a_{kl} for lucidity according to the numbers of the first indices, k, and then according to the second indices, l, we obtain a system of $n - 1$ homogeneous differential equations of the first order, which can be solved.

$$\left.\begin{aligned}
\frac{\mathrm{d}y_{12}}{\mathrm{d}x} &= -\Big(\sum_{k=2}^{n} a_{1k} + \sum_{k=1}^{1} a_{2k}\Big) y_{12} - \sum_{k=3}^{n} (a_{1k} - a_{2k})\, y_{23} - \dots \sum_{k=n}^{n} (a_{1k} - a_{2k})\, y_{n-1,n}, \\
\frac{\mathrm{d}y_{23}}{\mathrm{d}x} &= \sum_{k=1}^{1} (a_{2k} - a_{3k})\, y_{12} - \Big(\sum_{k=3}^{n} a_{2k} + \sum_{k=1}^{2} a_{3k}\Big) y_{23} - \dots \sum_{k=n}^{n} (a_{2k} - a_{3k})\, y_{n-1,n}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{\mathrm{d}y_{n-1,n}}{\mathrm{d}x} &= \sum_{k=1}^{1} (a_{n-1,k} - a_{nk})\, y_{12} + \sum_{k=1}^{2} (a_{n-1,k} - a_{nk})\, y_{23} + \dots - \Big(\sum_{k=n}^{n} a_{n-1,k} + \sum_{k=1}^{n-1} a_{nk}\Big) y_{n-1,n}.
\end{aligned}\right\} \quad (14)$$

This system contains one equation less than system (3) for the functions $t_1, t_2, \dots t_n$. However, the constant coefficients at $y_{12}, y_{23}, \dots y_{n-1,n}$ are more complicated than before.

For system (14) we obtain the following characteristic equation of the degree $n-1$ for r_i

$$\begin{vmatrix}
-\Big(r_i + \sum\limits_{k=2}^{n} a_{1k} + \sum\limits_{k=1}^{1} a_{2k}\Big) & -\sum\limits_{k=3}^{n} (a_{1k} - a_{2k}) \dots & -\sum\limits_{k=n}^{n} (a_{1k} - a_{2k}) \\
\sum\limits_{k=1}^{1} (a_{2k} - a_{3k}) & -\Big(r_i + \sum\limits_{k=3}^{n} a_{2k} + \sum\limits_{k=1}^{2} a_{3k}\Big) \dots & -\sum\limits_{k=n}^{n} (a_{2k} - a_{3k}) \\
\dots & \dots & \dots \\
\sum\limits_{k=1}^{1} (a_{n-1,k} - a_{nk}) & \sum\limits_{k=1}^{2} (a_{n-1,k} - a_{nk}) \dots & -\Big(r_i + \sum\limits_{k=n}^{n} a_{n-1,k} + \sum\limits_{k=1}^{n-1} a_{nk}\Big)
\end{vmatrix} = 0. \quad (15)$$

Assuming now *a priori* that this equation has simple roots only, we obtain the general solution in the form

$$\left.\begin{aligned}
y_{12} &= C_1\, \mathscr{D}_1^{(1)} e^{r_1 x} + C_2 \mathscr{D}_1^{(2)} e^{r_2 x} + \dots C_{n-1}\, \mathscr{D}_1^{(n-1)} e^{r_{n-1} x}, \\
y_{23} &= C_1\, \mathscr{D}_2^{(1)} e^{r_1 x} + C_2\, \mathscr{D}_2^{(2)} e^{r_2 x} + \dots C_{n-1}\, \mathscr{D}_2^{(n-1)} e^{r_{n-1} x}, \\
&\dots\dots\dots\dots\dots\dots \\
y_{k-1,k} &= C_1\, \mathscr{D}_{k-1}^{(1)} e^{r_1 x} + C_2\, \mathscr{D}_{k-1}^{(2)} e^{r_2 x} + \dots C_{n-1}\, \mathscr{D}_{k-1}^{(n-1)} e^{r_{n-1} x}, \\
y_{k,k+1} &= C_1\, D^{(1)} e^{r_1 x} + C_2\, D^{(2)} e^{r_2 x} + \dots C_{n-1}\, D^{(n-1)} e^{r_{n-1} x}, \\
y_{k+1,k+2} &= C_1\, \mathscr{D}_{k+1} e^{r_1 x} + C_2\, \mathscr{D}_{k+1}^{(2)} e^{r_2 x} + \dots C_{n-1}\, \mathscr{D}_{k+1}^{(n-1)} e^{r_{n-1} x}, \\
&\dots\dots\dots\dots\dots\dots \\
y_{n-1,n} &= C_1\, \mathscr{D}_{n-1}^{(1)} e^{r_1 x} + C_2\, \mathscr{D}_{n-1}^{(2)} e^{r_2 x} + \dots C_{n-1}\, \mathscr{D}_{n-1}^{(n-1)} e^{r_{n-1} x}.
\end{aligned}\right\} \quad (16)$$

In this solution $C_1, C_2, \dots C_{n-1}$ are of course constants which can be obtained by making use of the boundary conditions. $D^{(i)}$ and $\mathscr{D}_1^{(i)}, \mathscr{D}_2^{(i)}, \dots \mathscr{D}_{k-1}^{(i)}, \mathscr{D}_{k+1}^{(i)} \dots \mathscr{D}_{n-1}^{(i)}$ are as in solution (10), the relevant determinants of the order $n-2$, obtained from determinant (15) of the order $n-1$.

Since the third column and the third row are not written in expression (15), because of the lack of space, it should be observed that the formulae determining the elements of determinant (15) are regular on both sides of the main diagonal but in a different way. The formulae for the elements of the main diagonal are also of a different character and are regular.

The solution of equations (2a) can also be obtained in another way. By using the definition $y_{ij} = t_i - t_j$ we have for example

$$y_{ij} = y_{in} - y_{jn} \quad (17)$$

Hence it is easy to observe in equation (2), especially in the form of equation (2a), that any of the equations for dy_{ij}/dx with $i+1<j<n$ constitutes the difference between the equation for dy_{in}/dx and that for dy_{jn}/dx. Therefore the equations with $j<n$ do not constitute relations other than those expressed by the equations with the second index $j=n$ and can be omitted. Thus the solution of system (2a) can be reduced to the solution of the system

$$\left.\begin{aligned}
&\frac{dy_{1n}}{dx}+\sum_{k=1}^{n} a_{1k}\, y_{1k}-\sum_{k=1}^{n} a_{nk}\, y_{nk}=0,\\
&\frac{dy_{2n}}{dx}+\sum_{k=1}^{n} a_{2k}\, y_{2k}-\sum_{k=1}^{n} a_{nk}\, y_{nk}=0,\\
&\dots\dots\dots\dots\dots\dots\\
&\frac{dy_{n-1,n}}{dx}+\sum_{k=1}^{n} a_{n-1,k}\, y_{n-1,k}-\sum_{k=1}^{n} a_{nk}\, y_{nk}=0,
\end{aligned}\right\} \quad (18)$$

and some simple calculations of the remaining functions y_{ij} with $j<n$ according to formula (17).

System (18) is also indefinite and contains $n-1$ differential equations constituting the nth column of matrix (2) excluding the last equation on the main diagonal. We can transform this system of equations into a definite system with $n-1$ unknown functions y_{in} $(=y_{1n}, y_{2n}, \dots y_{n-1,n})$. For this purpose we express each one of equations (18) using formula (17) and bearing in mind that $y_{ij}=-y_{ji}$, $y_{ii}=0$ and $a_{ii}=0$.

$$\frac{dy_{in}}{dx}+\sum_{k=1}^{n} a_{ik}\, y_{ik}-\sum_{k=1}^{n} a_{nk}\, y_{nk}$$

$$=\frac{dy_{in}}{dx}+(a_{i1}\, y_{i1}+a_{i2}\, y_{i2}+\dots a_{ii}\, y_{ii}+\dots a_{i,n-1}\, y_{i,n-1}+a_{in}\, y_{in})$$

$$-(a_{n1}\, y_{n1}+a_{n2}\, y_{n2}+\dots a_{ni}\, y_{ni}+\dots a_{n,n-1}\, y_{n,n-1}+a_{nn}\, y_{nn})=$$

on introducing equation (17),

$$=\frac{dy_{in}}{dx}$$

$$+a_{i1}(y_{in}-y_{1n})+a_{i2}(y_{in}-y_{2n})+\dots a_{ii}(y_{in}-y_{in})+\dots a_{i,n-1}(y_{in}-y_{n-1,n})+a_{in}(y_{in}-y_{nn})$$

$$+\quad a_{n1}\, y_{1n}+\quad a_{n2}\, y_{2n}+\dots\quad a_{ni}\, y_{in}+\dots\quad a_{n,n-1}\, y_{n-1,n}+\quad a_{nn}\, y_{nn}.$$

Next ordering according to $y_{1n}, y_{2n}, \dots y_{n-1,n}$, we have

$$\left.\begin{aligned}
&\frac{dy_{in}}{dx}+\sum_{k=1}^{n} a_{ik}\, y_{ik}-\sum_{k=1}^{n} a_{nk}\, y_{nk}\\
&=\frac{dy_{in}}{dx}+(a_{n1}-a_{i1})\, y_{1n}+(a_{n2}-a_{i2})\, y_{2n}+\dots(a_{ni}+\sum_{k=1}^{n} a_{ik})y_{in}+\dots(a_{n,n-1}-a_{i,n-1})\, y_{n-1,n}.
\end{aligned}\right\} \quad (19)$$

Now we express equations (18) by means of formula (19), thus obtaining the following system of $n-1$ homogeneous differential equations of the first order determining the functions y_{1n}, y_{2n}, $\dots y_{n-1,n}$

$$\left.\begin{aligned}
\frac{dy_{1n}}{dx} &= -(\sum_{k=1}^{n} a_{1k} + a_{n1})\, y_{1n} + (a_{12} - a_{n2})\, y_{2n} + \ldots (a_{1,n-1} - a_{n,n-1})\, y_{n-1,n},\\
\frac{dy_{2n}}{dx} &= (a_{21} - a_{n1})\, y_{1n} - (\sum_{k=1}^{n} a_{2k} + a_{n2})\, y_{2n} + \ldots (a_{2,n-1} - a_{n,n-1})\, y_{n-1,n},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\frac{dy_{n-1,n}}{dx} &= (a_{n-1,1} - a_{n1})\, y_{1n} + (a_{n-1,2} - a_{n2})\, y_{2n} + \ldots - (\sum_{k=1}^{n} a_{n-1,k} + a_{n,n-1})\, y_{n-1,n}.
\end{aligned}\right\} \quad (20)$$

This system of equations has the following characteristic equation of the degree $n - 1$ for r_i

$$\begin{vmatrix}
-(r_i + \sum_{k=1}^{n} a_{1k} + a_{n1}) & a_{12} - a_{n2} & \ldots & a_{1,n-1} - a_{n,n-1}\\
a_{21} - a_{n1} & -(r_i + \sum_{k=1}^{n} a_{2k} + a_{n2}) & \ldots & a_{2,n-1} - a_{n,n-1}\\
\ldots & \ldots & \ldots & \ldots\\
a_{n-1,1} - a_{n-1} & a_{n-1,2} - a_{n2} & \ldots & -(r_i + \sum_{k=1}^{n} a_{n-1,k} + a_{n,n-1}
\end{vmatrix} = 0. \quad (21)$$

Assuming also *a priori* that this equation has simple roots only we obtain the general solution for $y_{1n}, y_{2n}, \ldots y_{n-1,n}$ in the same form as equation (16) for $y_{12}, y_{23}, \ldots y_{n-1,n}$.

By substituting, for instance,

$$y_{ij} = y_{i1} - y_{j1} \quad (22)$$

which is also in agreement with the definition $y_{ij} = t_i - t_j$ we obtain a system of $n - 1$ equations for $y_{12}, y_{13}, \ldots y_{1n}$. It is easy to prove that this system has the characteristic equation

$$\begin{vmatrix}
-(r_i + \sum_{k=1}^{n} a_{2k} + a_{12}) & a_{23} - a_{13} & \ldots & a_{2n} - a_{1n}\\
a_{32} - a_{12} & -(r_i + \sum_{k=1}^{n} a_{3k} + a_{13}) & \ldots & a_{3n} - a_{1n}\\
\ldots & \ldots & \ldots & \ldots\\
a_{n2} - a_{12} & a_{n3} - a_{13} & \ldots & -(r_i + \sum_{k=1}^{n} a_{nk} + a_{1n})
\end{vmatrix} = 0. \quad (23)$$

Assuming again that this equation has simple roots only, we obtain the general solution for $y_{12}, y_{13}, \ldots y_{1n}$ in the same form as equation (16) for $y_{12}, y_{23}, \ldots y_{n-1,n}$.

The above consideration concerning the equations for y_{ij} enable us to find the form of the determinant in the characteristic equations and the form of general solution. Moreover it has been shown that it is sufficient to choose $n - 1$ equations among the $(n^2 - n)/2$ equations constituting the elements of square matrix above the main diagonal. We have confined ourselves to the case of equations belonging to the first upper diagonal, the nth column and the first row. If equations of the matrix are designated by points, then the system mentioned above will lie on the sides of the rectangular triangle represented by continuous lines in Fig. 3. The results for the analogous triangle below the main diagonal will of course be identical. This conclusion is evident since $y_{ij} = -y_{ji}$ and

$$\left.\begin{aligned} y_{ji} &= y_{j,j-1} + y_{j-1,j-2} + \ldots y_{i+1,i} = -(y_{i,i+1} + y_{i+1,i+2} + \ldots y_{j-1,j}) = -y_{ij}, \\ y_{ji} &= y_{jn} - y_{in} = -(y_{in} - y_{jn}) \quad = -y_{ij}, \\ y_{ji} &= y_{j1} - y_{i1} = -(y_{i1} - y_{j1}) \quad = -y_{ij}. \end{aligned}\right\} \quad (24)$$

It can be observed that the application of the system of equations belonging to the arbitrary column l or the arbitrary row k leads also to a definite system of $n - 1$ equations. Then, it is necessary to use substitution of equations (17) and (19) with l instead of n or the substitution of equation (22) and another one with k instead of 1. Furthermore changing the functions y_{ji} into $-y_{ij}$ we obtain a right-angle rotation of the equation-line about the intersection point with the main diagonal. This is shown in Fig. 3 for the column l by means of a dotted line and corresponds to the solution in the form of $n - 1$ functions $y_{1l}, y_{2l}, \ldots y_{l-1,l}, y_{l,l+1}, \ldots y_{ln}$.

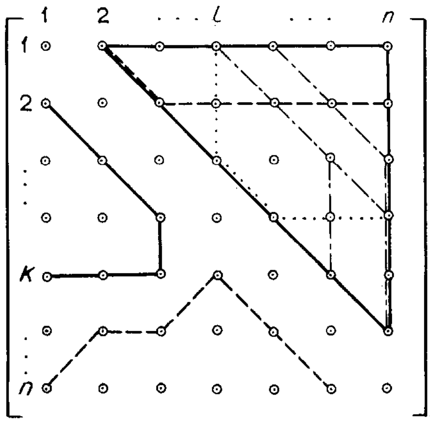


FIG. 3. Schematic drawing of matrix of equations determining y_{ij} and y_{ji} and configuration of the systems of equations selected from this matrix.

Let us observe now, that it is possible to use equations located on other diagonals, not neighbouring with the main diagonal. However, these must be supplemented by other equations located on the segment of any column or row between the diagonal under consideration and the main diagonal.

Many different systems of equations can be chosen from the matrix of equation (2). They correspond to various lines. Some of them are represented by a dashed line in Fig. 3. The form of the corresponding equations and determinants may be more irregular than those obtained above.

The above observations lead to the following conclusion of practical nature.

A definite system of equations is determined by a straight, angular or ramified line, formed by $n - 1$ elements of equation-matrix (2), *which has common points for each of the sides of the above triangle.*

The vertices of this triangle are treated of course as common points of two legs.

In the case where this condition is not satisfied the set of equations is indefinite and $(n^2 - n)/2$ functions y_{ij} or y_{ji} cannot be expressed in terms of $n - 1$ functions chosen from them. This occurs for example in the case illustrated in Fig. 3 by the angular continuous line in the region below the main diagonal and does not take place in the case described by the angular dashed line.

These observations may be used to the solution of practical problems with reduced number of heat exchanges between channels.

4. RELATIONS BETWEEN THE CHARACTERISTIC EQUATIONS OF THE SYSTEM OF EQUATIONS FOR t_k AND y_{ij}

In the general case of heat exchanger with the maximum possible number of heat exchanges the channels designated by 1, 2, . . . n can be ordered in any way with no influence on the form of the determinants. However, in practical cases with reduced number of heat exchanges the ordering of the labels in a certain order results in simpler determinants and more easy calculations. The ordering of the labels can be done according to various rules. For example we can attach successive numbers to the channels in the upward direction as is shown on the left-hand side of Fig. 1. Then, we can use the equations located on the continuous lines in Fig. 4 for y_{ij} and on the dashed lines for y_{ji}. The equations chosen in such a manner will concern only heat exchanging channels and determine the distribution of the temperature difference between these channels. It is these functions that are sought for in practical calculations of the mean temperature difference and exchanger efficiency.

It is worth mentioning that a slight change in the numerical notations of the channels may cause an essential change in the location of the line determining the system of equations chosen from matrix (2). For example in the case of multiloop heat exchanger a change from the notations on the left-hand side of Fig. 1(d) to those on the right-hand side will require the application of the equations in the last column of equation matrix (2) instead of those of the first row of this matrix.

Figure 4 indicates that all the practical cases of parallel-flow multichannel heat exchangers illustrated by Fig. 1 can be solved by using two groups of equations that of the first row or the first side diagonal just above the main diagonal of matrix (2).

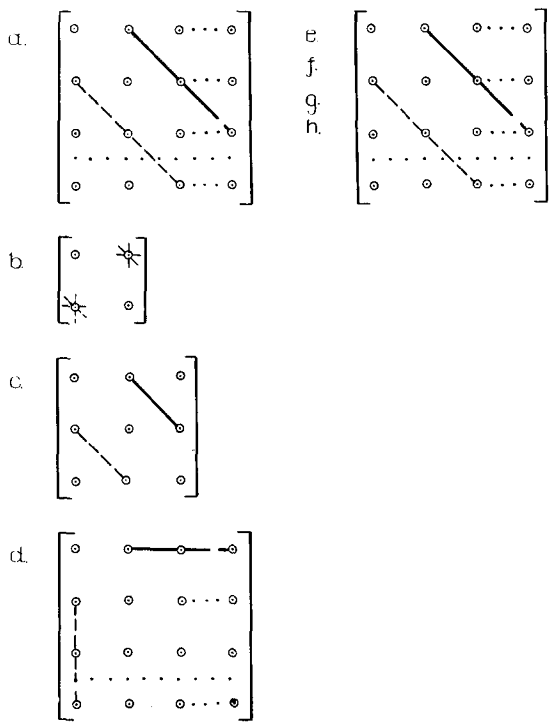


FIG. 4. Various schemes of the configuration of the systems of equations, determining y_{ij} and y_{ji}, of more important types for parallel-flow multichannel heat exchangers shown in Fig. 1.

These two groups of equations for y_{ij} correspond two variants of system of equations for t_k. Each of the latter has the same general solution but the coefficients of t_k becomes zero for different t_k. Hence the different form of the determinants giving the characteristic equations. These forms constitute particular, simplified cases of determinant (6).

Thus for the first variant concerning multi-loop heat exchangers only, of which the layout and the notations are as shown in the left-hand side of Fig. 1, we obtain from determinant (6) bearing in mind the fact that a_{kl} for $k = 1$ and $l = 2, 3, \ldots n$ or for $k = 2, 3, \ldots n$ and $l = 1$ are the only terms different from zero.

$$\begin{vmatrix} -(r_i + \sum_{k=2}^{n} a_{1k}) & a_{12} & a_{13} \ldots & a_{1n} \\ a_{21} & -(r_i + a_{21}) & 0 \ldots & 0 \\ a_{31} & 0 & -(r_i + a_{31}) \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & 0 & 0 \ldots & (-r_i + a_{n1}) \end{vmatrix} = 0. \quad (25)$$

We can reduce the order of this determinant by one by separating the factor r_i. For this we must perform the following operations:

We add to column 1 all the remaining columns.

We subtract row 1 from every remaining row.

We expand the determinant in minors according to the first column.

Then, since $(-1)^3 \neq 0$ we obtain

$$r_i \begin{vmatrix} -(r_i + a_{21} + a_{12}) & -a_{13} \ldots & -a_{1n} \\ -a_{12} & -(r_i + a_{31} + a_{13}) \ldots & -a_{1n} \\ \cdots & \cdots & \cdots \\ -a_{12} & -a_{13} \ldots & -(r_i + a_{n1} + a_{1n}) \end{vmatrix} = 0. \quad (26)$$

Now it is easy to prove that determinant (26), of the order $n - 1$, is the determinant that may be obtained in this particular case of multiloop heat exchanger from determinant (23).

The second variant of equations for t_k describes all the remaining types of heat exchangers under consideration. These are wave, screw, spiral, Field and ordinary parallel flow heat exchangers. Bearing in mind the fact that all the specific heat transferences a_{ii} and a_{ij}, a_{ji} with $j > i + 1$ are equal to zero, we obtain from determinant (6)

$$\begin{vmatrix} -(r_i + a_{12}) & a_{12} & 0 \ldots & 0 & 0 \\ a_{21} & -(r_i + a_{21} + a_{23}) & a_{23} \ldots & 0 & 0 \\ 0 & a_{32} & -(r_i + a_{32} + a_{34}) \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots & -(r_i + a_{n-1,n-2} + a_{n-1,n}) & a_{n-1,n} \\ 0 & 0 & 0 \ldots & a_{n,n-1} & -(r_i + a_{n,n-1}) \end{vmatrix} = 0. \quad (27)$$

The order of this determinant can also be reduced by one by separating the factor r_i before the determinant performing the following operations:

We add column 1 to column 2, resulting column 2 is added to column 3, etc. . . . the resulting column $n-1$ being added to the column n.

We subtract from row 1 row 2, from row 2 we subtract row 3, etc. . . . from the row $n-1$ being subtracted the row n.

We expand the determinant thus obtained in minors according to the last column.

Then since $(-1)^{2n+1} \neq 0$, we obtain

$$r_i \begin{vmatrix} -(r_i + a_{12} + a_{21}) & a_{23} & 0 & \dots & 0 \\ a_{21} & -(r_i + a_{23} + a_{32}) & a_{34} & \dots & 0 \\ 0 & a_{32} & -(r_i + a_{34} + a_{43}) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -(r_i + a_{n-1,n} + a_{n,n-1}) \end{vmatrix} = 0. \quad (28)$$

It is also easy to prove that determinant (28), of the order $n-1$, can be obtained from determinant (15) for the remaining particular cases of heat exchangers considered here.

The above results and the identity of determinants (6a) and (23) enable us to suppose that these constitute some particular cases of a general relation between the characteristic equations of the system of equations for t_k and y_{ij}. Indeed:

by adding column $n-1$ of determinant (23) to column $n-2$, column $n-2$ thus obtained to column $n-3$, etc. . . . column 2 thus obtained to column 1, cancelling then all the terms $a_{11}, a_{22} \dots a_{nn}$ $(=0)$,

subtracting now from row $n-1$ row $n-2$, from row $n-2$ row $n-3$, etc. . . . from row 2 row 1,

we obtain determinant (15).

Furthermore:

by adding row $n-1$ of determinant (15) to row $n-2$, the resulting row $n-2$ to row $n-3$, etc. . . . and resulting row 2 to row 1,

by subtracting column $n-2$ from column $n-1$, column $n-3$ from column $n-2$, etc. . . . and finally column 1 from column 2 and introducing $a_{11}, a_{22}, \dots a_{nn}$ $(=0)$ to the sums,

we obtain determinant (21).

Thus we can formulate the following:

Theorem 2.

For $a_{ii} = 0$ the roots of the characteristic equation are the same for each system of equations in y_{ij} and equal to the non-zero roots of the characteristic equation of the system of equations in t_k.

It follows that our *a priori* assumption of the existence of simple non-zero roots of the characteristic equations for y_{ij} is valid and that Theorem 1 concerns every possible group of equations for y_{ij}.

5. FINAL REMARKS

(1) The equality of values of determinants (6a), (15), (21) and (23) does not imply the equality of determinants $\mathscr{D}_1^{(i)}$, $\mathscr{D}_2^{(i)}$, . . . $\mathscr{D}_{k-1}^{(i)}$, $D^{(i)}$, $\mathscr{D}_{k+1}^{(i)}$, . . . $\mathscr{D}_{n-1}^{(i)}$ obtained by cancelling the columns with the same indices and the rows. Theorem 2 enables us to avoid only the necessity of solving the second characteristic equation if the problem is solved for verification purposes by two methods based on the system of equations for t_k and y_{ij}. It facilitates therefore this verification.

(2) In the solution of problems concerning heat exchangers with curved channels such as spiral or screw exchangers it is necessary to introduce the angular independent variable x.

Then, the specific heat transferences a_{kl} must be referred to this angular variable determining the length of the channel, and the boundary conditions must be expressed for definite radial planes such as it is shown in Fig. 1e, f.

(3) The present considerations enable us to reduce the solution of any problem of temperature distribution in channels of parallel-flow heat exchangers to the solution of algebraic equations. Having the general solution we avoid once and for all the derivation and solution of differential equations determining the temperature distribution. Moreover, the computation of complicated heat exchangers requires considerable labour as follows:

1. Numerical solution of algebraic equation of the order $n - 1$.
2. Expansion of n or $n - 1$ determinants of the order $n - 1$ or $n - 2$ and calculation of their n^2 or $(n - 1)^2$ values for various r_i.
3. Reordering the set of n or $n - 1$ algebraic equations for given boundary conditions.
4. Expansion and calculation of $n + 1$ or n determinants of the order n or $n - 1$ for the determination of the constants C.
5. Computation of the distribution of the temperature or the temperature difference.

Therefore for problems with more than five heat-exchanging channels it is reasonable to use an electronic computer or to elaborate a rapid approximate method. Further work will be devoted to the development of this method.

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Résumé—L'article présente la solution générale des systèmes d'équations différentielles décrivant les répartitions de la température et de la différence de température dans les canaux d'un échangeur. On a obtenu ces équations dans un autre article [13]. On trouve que pour n canaux le nombre d'équations est au plus n or $n - 1$. Les solutions sont données sous une forme permettant d'introduire les conditions aux limites. Ces solutions sont obtenues dans le cas général de canaux formant un faisceau et échangeant de la chaleur conformément au principe du plus grand nombre possible d'échanges de chaleur entre les canaux et dans la plupart des cas pratiques dans lesquels le nombre d'échanges de chaleur est réduit.

Zusammenfassung—Die Arbeit gibt die allgemeine Lösung von Differentialgleichungssystemen für die Temperatur- und Temperaturdifferenzverteilung in Flüssigkeiten, die in Kanälen von Wärmeübertragern strömen. Diese Gleichungen wurden in einer anderen Arbeit [13] abgeleitet. Für n Kanäle ergab sich eine Anzahl von höchstens n oder $n - 1$ Gleichungen. Die Lösungen sind so wiedergegeben, dass Grenzbedingungen eingeführt werden können. Diese Lösungen wurden für den allgemeinen Fall erhalten, dass die Kanäle einem Bündel vereinigt sind und zwischen den Kanälen die grösstmögliche Wärme übertragen wird, und für die häufigsten praktischen Fälle, in denen die Wärmeübertragung geringer bleibt.

Аннотация—В статье приводится общее решение системы дифференциальных уравнений, описывающих распределение и разность температур теплоносителей в каналах теплобоменника. Эти уравнения выводятся в работе [13]. Установлено, что для n каналов наибольшее число уравнений n и $n - 1$. Решения представлены в форме, позволяющей использовать граничные условия. Эти решения получены для общего случая пучка каналов, в которых теплообмен происходит по принципу максимально возможного числа теплообменов между ними, а также для практических случаев, где число теплообменов уменьшено.